

ON THE UNCONDITIONAL UNIQUENESS FOR NLS IN H^s ZHENG HAN¹ AND DAOYUAN FANG¹

ABSTRACT. In this article, we study the unconditional uniqueness of \dot{H}^s , $0 < s < 1$, solutions for the nonlinear Schrödinger equation $i\partial_t u + \Delta u + c|u|^\alpha u = 0$ in \mathbb{R}^n . We give a unified proof of the previously known results in the subcritical cases and critical cases, and we also extend these results to some previously unsettled cases. Our proof uses in particular negative order Sobolev spaces (or Besov spaces), general Strichartz estimates, and the improved regularity property for the difference of two solutions.

1. INTRODUCTION

We study the uniqueness of \dot{H}^s solutions of the following Schrödinger equation:

$$\begin{cases} i\partial_t u + \Delta u + c|u|^\alpha u = 0, \\ u(0) = \varphi \in \dot{H}^s(\mathbb{R}^n), \quad t \in [0, T], \quad x \in \mathbb{R}^n, \quad n \geq 2, \end{cases} \quad (\text{NLS})$$

where \dot{H}^s is the homogeneous Sobolev space, $c \in \mathbb{C}$, $T > 0$, $\alpha > 0$ and $s \in (0, \frac{n}{2})$.

To ensure that the initial value problem is locally well-posed in $H^s(\mathbb{R}^n)$, from Sobolev embedding, one has to assume $\alpha \leq \frac{4}{n-2s}$. Furthermore, the equation (NLS) may not make sense, even in the sense of distribution, without an auxiliary space if $\alpha > \frac{n+2s}{n-2s}$. Therefore, one usually constructs the solution within the framework of $C([0, T]; H^s) \cap X$, where X is an auxiliary space. For instance, Ginibre and Velo ([11]), Kato ([12]), Cazenave and Weissler ([4]) proved that (NLS) is locally well-posed in

$$C([0, T_{\max}); H^s) \cap L_{\text{loc}}^q(0, T_{\max}; B_{r,2}^s), \quad (1.1)$$

where $q = \frac{4(\alpha+2)}{\alpha(n-2s)}$, $r = \frac{n(\alpha+2)}{n+s\alpha}$ and $B_{r,2}^s$ is the usual Besov space.

The uniqueness of solutions that belongs to an auxiliary space such as $L_t^q B_{r,2}^s$ as well as $C([0, T]; H^s)$ is called conditional uniqueness. On the other hand, the uniqueness without any auxiliary space is called unconditional uniqueness. This problem, in the subcritical case, was first studied by Kato [12], in which the following results are obtained:

The uniqueness holds in $C([0, T]; H^s)$ if any of the following three conditions is satisfied:

- (1) $n = 1$, $0 \leq s < \frac{1}{2}$, $0 < \alpha < \frac{1+2s}{1-2s}$;
- (2) $n \geq 2$, $0 \leq s < \frac{n}{2}$, $0 < \alpha < \min\{\frac{4}{n-2s}, \frac{2+2s}{n-2s}\}$;
- (3) $n \geq 1$, $s \geq \frac{n}{2}$.

From Kato's work, we can see that when $1 \leq s$ and $0 < \alpha < \min\{\frac{4}{n-2s}, \frac{n+2s}{n-2s}\}$, the unconditional uniqueness holds.

2010 *Mathematics Subject Classification*. Primary: 35Q55, Secondary: 42E35.

Key words and phrases. Schrödinger equation, unconditional uniqueness, negative order Sobolev/Besov spaces, nonhomogeneous Strichartz estimates.

Furioli and Terraneo [10] extended Kato's results by using negative order Besov spaces. They proved uniqueness in the slightly larger space $C([0, T]; \dot{H}^s)$ when

$$n \geq 3, \max\{1, \frac{2s}{n-2s}\} < \alpha < \min\{\frac{2+4s}{n-2s}, \frac{4}{n-2s}, \frac{n+2s}{n-2s}, \frac{n+2-2s}{n-2s}\}. \quad (1.2)$$

In [17], Rogers applied a generalized Strichartz estimate (see [21]) to show that if

$$n \geq 3, \frac{2+2s}{n-2s} \leq \alpha < \min\{\frac{2+4s(1-\frac{1}{n})}{n-2s}, \frac{4}{n-2s}\}, \quad (1.3)$$

then uniqueness is established in $C([0, T]; \dot{H}^s)$.

Recently, Win and Tsutsumi [23] improved unconditional uniqueness in the dimension 3 under the following assumptions:

$$n = 3, 1 > s > \frac{1}{2}, \max\{\frac{2+4s(1-\frac{1}{n})}{n-2s}, \frac{n+2-2s}{n-2s}\} \leq \alpha < \min\{\frac{4}{n-2s}, \frac{n+2s}{n-2s}\}, \quad (1.4)$$

where the initial datum belongs to \dot{H}^s .

In summary, in the subcritical case, the problem of unconditional uniqueness is left open only when $0 \leq s < 1$, in the following three cases:

- Case a:** $n = 3, 4, \frac{2+4s(1-\frac{1}{n})}{n-2s} \leq \alpha < 1$ or $\max\{\frac{2+4s}{n-2s}, 1\} \leq \alpha < \min\{\frac{n+2s}{n-2s}, \frac{4}{n-2s}\}$;
- Case b:** $n = 5, \frac{2+4s(1-\frac{1}{n})}{n-2s} \leq \alpha \leq 1$ with $\alpha < \frac{4}{n-2s}$;
- Case c:** $n \geq 6, \frac{2+4s(1-\frac{1}{n})}{n-2s} \leq \alpha < \frac{4}{n-2s}$.

For the critical case $\alpha = \min\{\frac{n+2s}{n-2s}, \frac{4}{n-2s}\}$, we can recall the known results as follows: Kato firstly proved unconditional uniqueness in the dimension 1 or when $s \geq \frac{n}{2}$ in [12]. Cazenave [3](Proposition 4.2.13) showed that when $1 \leq s < n/2$ with $n \geq 3$, unconditional uniqueness still holds. Win and Tsutsumi [23] proved unconditional uniqueness in the following cases:

$$n = 3, 1 > s > \frac{1}{2}, \alpha = \frac{4}{n-2s} \text{ and } n = 4, 5, 1 > s \geq \frac{1}{2}, \alpha = \frac{4}{n-2s}. \quad (1.5)$$

There are also some gaps for the critical case, especially when $0 < s < \frac{1}{2}$ or high dimensions. In particular, for $0 \leq s < 1$, the following cases are open:

- Case a:** $n = 2, \alpha = \frac{n+2s}{n-2s}$;
- Case b:** $n = 3, 0 \leq s \leq \frac{1}{2}$ with $\alpha = \frac{n+2s}{n-2s}$;
- Case c:** $n = 4, 5, 0 \leq s < \frac{1}{2}$ with $\alpha = \frac{4}{n-2s}$;
- Case d:** $n \geq 6, \alpha = \frac{4}{n-2s}$.

From the above description, the authors in [12],[10],[17] and [23] apply different methods to obtain various conclusions. The conclusions they obtained overlap, but do not cover each other. In this article, in addition to extending the known results to a larger domain of indices, in particular the case $\alpha < 1$, we also give a unified proof of the results of [10],[17],[23] and [12] in either subcritical case or critical case. Note that the nonlinearity is locally Lipschitz continuous when $\alpha \geq 1$, and locally Hölder continuous when $\alpha < 1$. For this reason, we have to use the different argument in cases $0 \leq \alpha < 1$ and $\alpha \geq 1$. Firstly, we show the results on the subcritical case:

Theorem 1.1 ($\alpha \geq 1$). *Let $0 < s < 1$, $n = 3, 4, 5$, and assume*

$$\max\{1, \frac{2s}{n-2s}\} \leq \alpha < \min\{\frac{4}{n-2s}, \frac{n+2s}{n-2s}, \frac{4s+4-n/(n-1)}{n-2s}\}. \quad (1.6)$$

Given $\varphi \in \dot{H}^s$ and $T > 0$, unconditional uniqueness holds in $L^\infty(0, T; \dot{H}^s)$ for (NLS).

Theorem 1.2 ($\alpha < 1$). *Let $0 < s < 1$, $\varphi \in \dot{H}^s$ and $T > 0$. If α satisfies one of the following conditions, then unconditional uniqueness holds in $L^\infty(0, T; \dot{H}^s)$ for (NLS):*

- when $n = 3$,

$$\begin{cases} \frac{2s}{3-2s} < \alpha < 1, & \text{if } s \geq \frac{3}{4}, \\ \frac{2s}{3-2s} < \alpha < \min\{1, \frac{2s+\frac{5}{2}}{3-4s}\}, & \text{if } s < \frac{3}{4}, \end{cases} \quad (1.7)$$

- when $n \geq 4$,

$$\frac{2s}{n-2s} < \alpha < \min\{1, \frac{4}{n-2s}, \frac{2s+4-\frac{n}{n-1}}{n-4s}\}. \quad (1.8)$$

Remark 1.3. According to our results, the following cases for unconditional uniqueness are still left open for $0 \leq s < 1$:

Case a: $n = 3, 4$, $\min\{\frac{4s+4-n/(n-1)}{n-2s}, \frac{2s+4-n/(n-1)}{n-4s}\} \leq \alpha < \min\{\frac{4}{n-2s}, \frac{n+2s}{n-2s}\}$;

Case b: $n \geq 5$, $\frac{2s+4-n/(n-1)}{n-4s} \leq \alpha < \frac{4}{n-2s}$;

Case c: $n \geq 3$, $s = 0$, $\frac{2}{n} \leq \alpha < \frac{4-n/(n-1)}{n}$.

Remark 1.4. It is not difficult to verify that the results of [10], [17] and [23] (for $n = 3$ and $\alpha < 4/(n-2s)$) are covered by Theorem 1.1 and Theorem 1.2. The conclusions of [12] are also included in our results when $\alpha > \frac{2s}{n-2s}$.

The strategy of our proof is similar to the one used by Furioli and Terraneo in [10], and it makes use of the negative order homogeneous Besov space $\dot{B}_{\rho,2}^\sigma$ and Sobolev space \dot{H}_ρ^σ respectively. For the choice of ρ , in addition to that used by Furioli and Terraneo in [10], we can also select different indices. Generally speaking, if $u, v \in L^\infty(I; \dot{H}^s)$ are two solutions of (NLS), in order that $u - v \in L^\infty(I; \dot{B}_{\rho,2}^\sigma)$, the relationship $s - \frac{n}{2} = \sigma - \frac{n}{\rho}$ is natural by the embedding $\dot{H}^s \hookrightarrow \dot{B}_{\rho,2}^\sigma$ with $\sigma < 0$. However, the difference of two solutions sometimes has better regularity in certain spaces than each of the solutions. We show this better regularity for the subcritical case in the Part 3.2 and for the critical case in the Part 4.3.1.

We also use nonhomogeneous Strichartz estimates, which are different from those used in [10]. Furioli and Terraneo applied the classical Strichartz estimates:

$$\left\| \int_0^t e^{i(t-s)\Delta} f(s) ds \right\|_{L^{q_1}(I; L^{r_1})} \leq C \|f\|_{L^{q_2'}(I; L^{r_2'})},$$

where $(e^{it\Delta})_{t \in \mathbb{R}}$ is the Schrödinger group and $f \in L^{q_2'}(I; L^{r_2'})$, and the pairs (q_i, r_i) , $i = 1, 2$ satisfy the admissibility conditions $\frac{2}{q_i} = n(\frac{1}{2} - \frac{1}{r_i})$ and $2 \leq r_i \leq 2n/(n-2)$ ($2 \leq r_i \leq \infty$ if $n = 1$, $2 \leq r_i < \infty$ if $n = 2$). Therefore, in order to make the selected ρ part of an admissible pair, the condition $s - 1 \leq \sigma \leq s$ should be satisfied. Furthermore, Furioli and Terraneo only settled the case $\alpha > 1$, where the nonlinearity is locally Lipschitz continuous. Their method does not apply to the case $\alpha < 1$, when the nonlinearity is not locally Lipschitz. We apply the general

Strichartz estimates, which are described in Lemma 2.1, ρ being restricted to be part of a “general” admissible pair. This improves the previous restriction on σ . Our restrictions on σ , for instance given by (3.7)-(3.8) or (3.13) when $\alpha \geq 1$ and (3.16)-(3.17) or (3.22)-(3.23) when $\alpha < 1$, corresponding to the different choices ρ . In fact, if $s \geq \frac{1}{2}$, $\alpha > \max\{1, \frac{2}{n-2s}\}$, in light of (3.7)-(3.8), we may choose $\sigma = s-1$, which is the choice made by Win and Tsutsumi in [23]. For the case $\alpha < 1$, we use the fractional chain rule for a Hölder continuous function (Lemma 2.4), then a result similar to Lemma 2.3 in [10] is obtained, which is applied to control the nonlinearity. From the proof of the case $\alpha \geq 1$, we can see that the bound $(4s+4-\frac{n}{n-1})/(n-2s)$ comes from the condition $\sigma+s \geq 0$, which ensures (3.5) to hold. A similar argument can be used in the case $\alpha < 1$.

We also consider the critical cases in the following results:

Theorem 1.5. *Let $\alpha = \frac{n+2s}{n-2s}$ and $n = 2$ with $0 < s < 1$ or $n = 3$ with $\frac{1}{4} < s < \frac{1}{2}$. Given $\varphi \in \dot{H}^s$ and $T > 0$, unconditional uniqueness holds in $L^\infty(0, T; \dot{H}^s)$ for (NLS).*

Theorem 1.6. *Let $\alpha = \frac{4}{n-2s}$ and $n = 3$ with $1/2 < s < 1$ or $n = 4$ with $1/3 < s < 1$ or $n \geq 5$ with $s_0 < s < 1$, where s_0 is the smallest solution of equation $4(n-1)s^2 - (2n^2 + 8n - 8)s + n^2 = 0$. Given $\varphi \in \dot{H}^s$ and $T > 0$, unconditional uniqueness holds in $C([0, T]; \dot{H}^s)$ for (NLS).*

Remark 1.7. It follows from Theorem 1.5 and 1.6 that unconditional uniqueness in the critical case is left open in the following cases:

- Case a:** $n = 2$, $\alpha = 1$ and $s = 0$;
- Case b:** $n = 3$, $\alpha = \frac{n+2s}{n-2s}$ and $0 \leq s \leq \frac{1}{4}$ or $s = \frac{1}{2}$;
- Case c:** $n = 4$, $\alpha = \frac{4}{n-2s}$ and $0 \leq s \leq \frac{1}{3}$;
- Case d:** $n \geq 5$, $\alpha = \frac{4}{n-2s}$ and $0 \leq s \leq s_0$.

Remark 1.8. Note that Theorem 1.5 states uniqueness of solutions in $L^\infty(0, T; \dot{H}^s)$, while Theorem 1.6 states uniqueness for solutions in a stronger sense, i.e. solutions in $C([0, T]; \dot{H}^s)$. The fundamental reason is that, under the assumptions of Theorem 1.5, when estimating the difference of two solutions, there comes a factor of T in the right-hand side. So we can choose T sufficiently small so that the right hand side is absorbed by the left hand side. However, under the assumptions of Theorem 1.6, the coefficient is no longer dependent on time. A similar difficulty appears in [3] and [23]. Using an argument inspired by [3, 23], we divide the nonlinearity by high-low frequencies and use the norms $L_t^\gamma \dot{H}_\rho^\sigma \cap L_t^a \dot{H}_b^\sigma$, where the parameters σ , γ , ρ , a and b are chosen in Section 4.

Notation: H^s is the Sobolev space and \dot{H}^s is the homogeneous Sobolev space, see Section 6.2 and Section 6.3 of [1] respectively. Similarly, $B_{p,q}^s$ and $\dot{B}_{p,q}^s$ are the Besov spaces and the homogeneous Besov spaces, as defined in Section 6.2 and Section 6.3 of [1].

The paper is organized as follows: in Section 2, we state and prove some preparatory lemmas; in Section 3, we give the proofs of Theorem 1.1 and Theorem 1.2; Section 4 is devoted to the proofs of Theorem 1.5 and Theorem 1.6. Finally, we present four figures at the end of the paper, displaying in dimensions $n = 3$, $n = 4$, $n = 5$ and $n \geq 6$, respectively, the various regions where unconditional uniqueness is known or is still an open problem.

2. PRELIMINARIES

In this section, we present some lemmas which we need. The first one is non-homogeneous Strichartz estimate which is due to Foschi [9]. This estimate extends results of Strichartz [18], Ginibre and Velo [11], Yajima [24], Cazenave and Weissler [5], Keel and Tao [14].

Definition 2.1. We say that the pair (q, r) is $\frac{n}{2}$ -acceptable if

$$1 \leq q < \infty, \quad 2 \leq r \leq \infty, \quad \frac{1}{q} < n(\frac{1}{2} - \frac{1}{r}), \quad \text{or } (q, r) = (\infty, 2). \quad (2.1)$$

Lemma 2.1 (Nonhomogeneous Strichartz estimate). *Given any $\sigma \in \mathbb{R}$, the following properties holds:*

Let I be an interval of \mathbb{R} , $J = \bar{I}$, and $0 \in J$. If (q, r) is a $\frac{n}{2}$ -acceptable pair and $f \in L^{q'}(I; \dot{H}_r^\sigma)$, then for every $\frac{n}{2}$ -acceptable pair (γ, ρ) , there exists a constant C independent of I such that

$$\left\| \int_0^t e^{i(t-s)\Delta} f(s) ds \right\|_{L^\gamma(I; \dot{H}_\rho^\sigma)} \leq C \|f\|_{L^{q'}(I; \dot{H}_r^\sigma)}, \quad (2.2)$$

when γ, ρ, q and r verify the scaling condition

$$\frac{1}{q} + \frac{1}{\gamma} = \frac{n}{2} \left(1 - \frac{1}{r} - \frac{1}{\rho}\right) \quad (2.3)$$

and satisfy one of the following sets of conditions:

- if $n = 2$, we also require that $r, \rho < \infty$;
- if $n \geq 3$, we distinguish two cases,

-non sharp case:

$$\frac{1}{q} + \frac{1}{\gamma} < 1, \quad (2.4)$$

$$\left(\frac{n}{2} - 1\right)\frac{1}{r} \leq \frac{n}{2\rho}, \quad \left(\frac{n}{2} - 1\right)\frac{1}{\rho} \leq \frac{n}{2r}; \quad (2.5)$$

-sharp case:

$$\frac{1}{q} + \frac{1}{\gamma} = 1, \quad (2.6)$$

$$\left(\frac{n}{2} - 1\right)\frac{1}{r} < \frac{n}{2\rho}, \quad \left(\frac{n}{2} - 1\right)\frac{1}{\rho} < \frac{n}{2r}, \quad (2.7)$$

$$\frac{1}{r} \leq \frac{1}{q}, \quad \frac{1}{\rho} \leq \frac{1}{\gamma}. \quad (2.8)$$

The Sobolev space can be replaced by Besov space, where the conditions $\gamma, q \geq 2$ have to hold.

Proof. The estimate without derivatives follows from [9]. The proof for the Sobolev spaces is simple if we notice the fact

$$e^{i(t-s)\Delta} [\mathcal{F}^{-1}(|\xi|^\sigma \hat{f})] = \mathcal{F}^{-1}[|\xi|^\sigma \mathcal{F}(e^{i(t-s)\Delta} f(s))], \quad (2.9)$$

where $(e^{it\Delta})_{t \in \mathbb{R}}$ is the Schrödinger group and \mathcal{F} is the Fourier transform.

For the case of Besov spaces, by the definition of the homogeneous Besov space (see section 6.3 of [1]), we have

$$\begin{aligned} & \left\| \int_0^t e^{i(t-s)\Delta} f(s) ds \right\|_{L^\gamma(I; \dot{B}_{\rho,2}^\sigma)}^2 \\ &= \left\| \left(\sum_{j=-\infty}^{\infty} (2^{\sigma j} \|\mathcal{F}^{-1}(\psi_j \mathcal{F}(\int_0^t e^{i(t-s)\Delta} f(s) ds))\|_{L^\rho})^2 \right)^{\frac{1}{2}} \right\|_{L^\gamma(I)}^2 \\ &:= \|\Phi_j\|_{L_I^\gamma L_j^2 L_x^\rho}^2, \end{aligned} \quad (2.10)$$

where $\mathcal{F}^{-1}\psi_j$ is the homogeneous dyadic decomposition. By Minkowski's inequality and estimate of [9], we have

$$\begin{aligned} \|\Phi_j\|_{L_I^\gamma L_j^2 L_x^\rho}^2 &\leq \|\Phi_j\|_{L_I^2 L_j^\gamma L_x^\rho}^2 \\ &\leq \|2^{\sigma j} \mathcal{F}^{-1}(\psi_j \hat{f})\|_{L_j^2 L_I^{q'} L_x^{r'}}^2 \leq \|2^{\sigma j} \mathcal{F}^{-1}(\psi_j \hat{f})\|_{L_I^{q'} L_j^2 L_x^{r'}}^2 \\ &= \|f\|_{L^{q'}(I; \dot{B}_{r',2}^\sigma)}^2 \end{aligned} \quad (2.11)$$

if $\gamma, q \geq 2$, which completes the proof. \square

For the Cauchy problem in H^s spaces, we cannot avoid to estimate the nonlinearity with some fractional derivative. Therefore, we need the fractional chain rule and bilinear estimate for the nonlinearity in Sobolev space and Besov space.

Lemma 2.2 (Product rule). *Let $s \in (0, 1)$ and $1 < r, p_1, p_2, q_1, q_2 < \infty$ such that $\frac{1}{r} = \frac{1}{p_i} + \frac{1}{q_i}$ for $i = 1, 2$. If $f \in L^{p_1} \cap \dot{H}_{p_2}^s \cap \dot{B}_{p_2,2}^s$ and $g \in L^{q_2} \cap \dot{H}_{q_1}^s \cap \dot{B}_{q_1,2}^s$, then*

$$\| |\nabla|^s (fg) \|_{L^r} \lesssim \|f\|_{L^{p_1}} \| |\nabla|^s g \|_{L^{q_1}} + \|g\|_{L^{q_2}} \| |\nabla|^s f \|_{L^{p_2}}, \quad (2.12)$$

$$\|fg\|_{\dot{B}_{r,2}^s} \lesssim \|f\|_{L^{p_1}} \|g\|_{\dot{B}_{q_1,2}^s} + \|g\|_{L^{q_2}} \|f\|_{\dot{B}_{p_2,2}^s}. \quad (2.13)$$

Proof. Estimate (2.12) follows from Proposition 3.3 of [6]. For the case of Besov space, using the equivalence of the norm (see theorem 6.3.1 in [1]), we have

$$\|fg\|_{\dot{B}_{r,2}^s} = \left(\int_0^\infty (t^{-s} \sup_{|y| \leq t} \|(fg)(\cdot - y) - (fg)(\cdot)\|_{L^r})^2 \frac{dt}{t} \right)^{\frac{1}{2}}. \quad (2.14)$$

Note that

$$(fg)(\cdot - y) - (fg)(\cdot) = (f(\cdot - y) - f(\cdot))g(\cdot - y) + (g(\cdot - y) - g(\cdot))f(\cdot), \quad (2.15)$$

then by Hölder inequality and (2.14), we can show that (2.13) is true. \square

Lemma 2.3 ([6]). *Suppose $G \in C^1(\mathbb{C})$, $s \in (0, 1]$, and $1 < p, p_1, p_2 < \infty$ are such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then,*

$$\| |\nabla|^s G(u) \|_{L^p} \lesssim \|G'(u)\|_{L^{p_1}} \| |\nabla|^s u \|_{L^{p_2}}. \quad (2.16)$$

Lemma 2.4 (Fractional chain rule for a Hölder continuous function, Proposition A.1 in [22]). *Let G be a Hölder continuous function of order $0 < \alpha < 1$. Then, for every $0 < s < \alpha$, $1 < p < \infty$, and $\frac{s}{\alpha} < \sigma < 1$, we have*

$$\| |\nabla|^s G(u) \|_{L^p} \lesssim \| |u|^{\alpha - \frac{s}{\sigma}} \|_{L^{p_1}} \| |\nabla|^\sigma u \|_{L^{\frac{s}{\sigma} p_2}}, \quad (2.17)$$

provided $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $(1 - \frac{s}{\alpha\sigma})p_1 > 1$.

Lemma 2.5. *Let $-1 < \sigma < 0$ and $1 < \rho, p_1, p_2, p_3, r < \infty$ such that $\frac{1}{\rho'} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{r}$ and $\frac{1}{p_2} = \frac{1}{r} + \frac{\sigma}{n}$. Then for any $f \in L^{p_3} \cap \dot{H}_{p_1}^{-\sigma}$ and $g \in \dot{H}_\rho^\sigma$, we have*

$$\|fg\|_{\dot{H}_{r'}^\sigma} \lesssim \|g\|_{\dot{H}_\rho^\sigma} (\|f\|_{\dot{H}_{p_1}^{-\sigma}} + \|f\|_{L^{p_3}}). \quad (2.18)$$

Furthermore, if $p_2 \geq 2$, then for any $f \in L^{p_3} \cap \dot{B}_{p_1,2}^{-\sigma}$ and $g \in \dot{B}_{\rho,2}^\sigma$, we have

$$\|fg\|_{\dot{B}_{r',2}^\sigma} \lesssim \|g\|_{\dot{B}_{\rho,2}^\sigma} (\|f\|_{\dot{B}_{p_1,2}^{-\sigma}} + \|f\|_{L^{p_3}}). \quad (2.19)$$

Proof. We only prove the case of Sobolev spaces, a similar argument can be used for the case of Besov spaces. By duality, to prove (2.18), we need only prove the following inequality

$$|\langle fg, h \rangle| \lesssim \|g\|_{\dot{H}_\rho^\sigma} (\|f\|_{\dot{H}_{p_1}^{-\sigma}} + \|f\|_{L^{p_3}}) \|h\|_{\dot{H}_r^{-\sigma}}, \quad (2.20)$$

where $\langle \cdot, \cdot \rangle$ denotes the L^2 scalar product.

By (2.12) of Lemma 2.2, Hölder inequality and Sobolev's embedding, it follows that

$$\begin{aligned} |\langle fg, h \rangle| &= |\langle g, \bar{f}h \rangle| \leq \|g\|_{\dot{H}_\rho^\sigma} \|\bar{f}h\|_{\dot{H}_{\rho'}^{-\sigma}} \\ &\lesssim \|g\|_{\dot{H}_\rho^\sigma} (\|f\|_{\dot{H}_{p_1}^{-\sigma}} \|h\|_{L^{p_2}} + \|f\|_{L^{p_3}} \|h\|_{\dot{H}_r^{-\sigma}}) \\ &\lesssim \|g\|_{\dot{H}_\rho^\sigma} (\|f\|_{\dot{H}_{p_1}^{-\sigma}} + \|f\|_{L^{p_3}}) \|h\|_{\dot{H}_r^{-\sigma}}. \end{aligned} \quad (2.21)$$

□

3. THE PROOF OF THEOREMS 1.1 AND 1.2

In this section, we give the proofs of Theorem 1.1 and Theorem 1.2. We invoke some negative order Sobolev (or Besov) spaces, the general nonhomogeneous Strichartz estimate and properties of the solutions to achieve our goal. Let u and v be two $L^\infty(0, T; \dot{H}^s)$ solutions of (NLS) with the same initial data φ and $T > 0$. The Parts 3.1 and 3.2 are devoted to the proof of the case $1 \leq \alpha$, and the rest illustrate the proof for $0 < \alpha < 1$. For the sake of simplicity, we denote $f(u) = c|u|^\alpha u$.

3.1. Usual regularity property case. We consider the space $L^\gamma(0, T; \dot{B}_{\rho,2}^\sigma)$ for certain $n/2$ -acceptable pair (γ, ρ) , with $\frac{1}{\rho} = \frac{\sigma}{n} + \frac{1}{2} - \frac{s}{n}$, $\sigma < 0$, where σ and γ can be fixed later. We say $u - v$ has usual regularity property if it belongs to the same auxiliary space $L^\gamma(0, T; \dot{B}_{\rho,2}^\sigma)$ as that u, v belong to, by embedding $\dot{H}^s \hookrightarrow \dot{B}_{\rho,2}^\sigma$ for $u, v \in L^\infty(0, T; \dot{H}^s)$ with finite time T . Our aim is to show the uniqueness in the space $L^\gamma(0, T; \dot{B}_{\rho,2}^\sigma)$.

By using Duhamel's formula and Lemma 2.1 in non sharp case, we have

$$\|u - v\|_{L^\gamma(0, T; \dot{B}_{\rho,2}^\sigma)} \lesssim \|f(u) - f(v)\|_{L^{q'}(0, T; \dot{B}_{r',2}^\sigma)}, \quad (3.1)$$

where $\frac{1}{r} = \frac{1}{2} - \frac{\sigma}{n} + \frac{s}{n} - \frac{n-2s}{2n}\alpha$, (q, r) is a $\frac{n}{2}$ -acceptable pair and γ, ρ, q and r satisfy the conditions (2.3)-(2.5) with $\gamma, q \geq 2$.

Given $u, v \in \mathbb{C}$, we have

$$\begin{aligned} f(u) - f(v) &= (u - v) \int_0^1 \partial_z f(v + \theta(u - v)) d\theta + \\ &\quad (\overline{u - v}) \int_0^1 \partial_{\bar{z}} f(v + \theta(u - v)) d\theta, \end{aligned}$$

or, in short,

$$f(u) - f(v) = (u - v) \int_0^1 f'(v + \theta(u - v)) d\theta. \quad (3.2)$$

If $\frac{1}{p_1} = \frac{n-2s}{2n}\alpha - \frac{\sigma}{n}$, $\frac{1}{p_3} = \frac{n-2s}{2n}\alpha$, σ and r satisfy the conditions of Lemma 2.5, then we have

$$\begin{aligned} & \|f(u) - f(v)\|_{\dot{B}_{r',2}^\sigma} \\ & \lesssim \|u - v\|_{\dot{B}_{\rho,2}^\sigma} \left(\int_0^1 \|f'(v + \theta(u - v))\|_{\dot{B}_{p_1,2}^{-\sigma}} d\theta + \int_0^1 \|f'(v + \theta(u - v))\|_{L^{p_3}} d\theta \right). \end{aligned} \quad (3.3)$$

By the form of f' and Sobolev embedding $\dot{H}^s \hookrightarrow L^{\frac{2n}{n-2s}}$, we see

$$\int_0^1 \|f'(v + \theta(u - v))\|_{L^{p_3}} d\theta \lesssim (\|u\|_{L^{p_3\alpha}} + \|v\|_{L^{p_3\alpha}})^\alpha \lesssim (\|u\|_{\dot{H}^s} + \|v\|_{\dot{H}^s})^\alpha. \quad (3.4)$$

Using the equivalent norm of Besov space (see theorem 6.3.1 in [1]), Hölder's inequality, the embedding $\dot{H}^s \hookrightarrow L^{\frac{2n}{n-2s}}$, $\dot{H}^s \hookrightarrow \dot{B}_{l,2}^{-\sigma}$, and $\alpha \geq 1$, we have

$$\begin{aligned} & \int_0^1 \|f'(v + \theta(u - v))\|_{\dot{B}_{p_1,2}^{-\sigma}} d\theta \\ & = \int_0^1 \left\{ \int_0^\infty t^{2\sigma} \left(\sup_{|y| \leq t} \|(f'(v + \theta(u - v)))_y - (f'(v + \theta(u - v)))\|_{L^{p_1}} \right)^2 \frac{dt}{t} \right\}^{\frac{1}{2}} d\theta \\ & \lesssim \int_0^1 \int_0^1 \left\{ \int_0^\infty t^{2\sigma} \left[\sup_{|y| \leq t} \|\eta(\theta u + (1 - \theta)v)_y + (1 - \eta)(\theta u + (1 - \theta)v)\|_{L^{\frac{2n}{n-2s}}}^{\alpha-1} \right. \right. \\ & \quad \left. \left. \|\theta u + (1 - \theta)v)_y - (\theta u + (1 - \theta)v)\|_{L^l} \right]^2 \frac{dt}{t} \right\}^{\frac{1}{2}} d\eta d\theta \\ & \lesssim (\|u\|_{L^{\frac{2n}{n-2s}}} + \|u\|_{L^{\frac{2n}{n-2s}}})^{\alpha-1} (\|u\|_{\dot{B}_{l,2}^{-\sigma}} + \|v\|_{\dot{B}_{l,2}^{-\sigma}}) \\ & \lesssim (\|u\|_{\dot{H}^s} + \|u\|_{\dot{H}^s})^\alpha, \end{aligned} \quad (3.5)$$

where $u(\cdot - y) := u_y$, $\frac{1}{l} = \frac{1}{2} - \frac{s}{n} - \frac{\sigma}{n}$ and $s \geq -\sigma$.

Then, by (3.3), (3.4), (3.5) and Hölder's inequality in time, it follows from (3.1) that

$$\|u - v\|_{L^\gamma(0,T;\dot{B}_{\rho,2}^\sigma)} \lesssim T^{1-\frac{1}{q}-\frac{1}{\gamma}} \left(\|u\|_{L^\infty(0,T;\dot{H}^s)} + \|v\|_{L^\infty(0,T;\dot{H}^s)} \right)^\alpha \|u - v\|_{L^\gamma(0,T;\dot{B}_{\rho,2}^\sigma)}. \quad (3.6)$$

Therefore, if T is sufficiently small, we have $u = v$ on $[0, T]$.

We summarize the conditions that we have imposed so far on the parameters $q, r, \gamma, \rho, \sigma$:

- (1) the choices of ρ and r : $\frac{1}{\rho} = \frac{\sigma}{n} + \frac{1}{2} - \frac{s}{n}$, $\frac{1}{r} = \frac{1}{2} - \frac{\sigma}{n} + \frac{s}{n} - \frac{n-2s}{2n}\alpha$;
- (2) (γ, ρ) , (q, r) being $\frac{n}{2}$ -acceptable pairs;
- (3) (γ, ρ) , (q, r) satisfying the conditions (2.3)-(2.5) and $\gamma, q \geq 2$;
- (4) conditions on σ and r for the validity of Lemma 2.5,

$$-1 < \sigma < 0, \quad 0 < \frac{1}{r} + \frac{\sigma}{n} \leq \frac{1}{2},$$

where the second is equivalent to $\frac{2s}{n-2s} \leq \alpha < \frac{n+2s}{n-2s}$;

(5) condition on σ for the validity of (3.5),

$$s \geq -\sigma.$$

From the conditions (1)-(5), the restrictions on σ are finally derived:

$$\max\left\{-s, s - \frac{n}{2(n-1)} - \frac{(n-2)(n-2s)}{4(n-1)}\alpha\right\} \leq \sigma \leq s + \frac{n}{2(n-1)} - \frac{n(n-2s)}{4(n-1)}\alpha, \quad (3.7)$$

$$\max\left\{s - \frac{1}{2} - \frac{n-2s}{4}\alpha, s - \frac{n-2s}{2}\alpha, s - \frac{n}{2}\right\} < \sigma < \min\left\{0, s + \frac{1}{2} - \frac{n-2s}{4}\alpha\right\}. \quad (3.8)$$

It follows from a simple calculation that the set consisting of the elements which satisfy the conditions (3.7)-(3.8) is non-empty if s, α and n satisfy the conditions

$$\begin{cases} 1 \leq \alpha < \min\left\{\frac{4}{n-2s}, \frac{n+2s}{n-2s}\right\}, \\ \frac{2s}{n-2s} < \alpha \leq (2 + 8s(1 - \frac{1}{n}))/ (n-2s), \\ 0 < s < 1 \text{ and } n = 3, 4, 5. \end{cases} \quad (3.9)$$

Therefore, under the conditions of (3.9), we establish the unconditional uniqueness of (NLS).

3.2. Better regularity property case. This part is devoted to the study of the cases

$$\begin{cases} \max\{1, (2 + 8s(1 - \frac{1}{n}))/ (n-2s)\} < \alpha < \min\left\{\frac{4}{n-2s}, \frac{n+2s}{n-2s}\right\}, \\ 0 < s < 1 \text{ and } n = 3, 4, 5. \end{cases} \quad (3.10)$$

We consider the space $L^\gamma(0, T; \dot{B}_{\rho, 2}^\sigma)$ for the $n/2$ -acceptable pair (γ, ρ) , with $\frac{1}{\rho} = \frac{\sigma}{n} + \frac{1}{2} - \frac{s}{n} + \frac{n-2s}{2n}\alpha - \frac{2}{n}$ and $\sigma < 0$. It is clear that u, v are no longer in $L^\gamma(0, T; \dot{B}_{\rho, 2}^\sigma)$ because of $\dot{H}^s \not\subset \dot{B}_{\rho, 2}^\sigma$. However, in some restricted conditions on ρ , we can show $u - v \in L^\gamma(0, T; \dot{B}_{\rho, 2}^\sigma)$.

Since $\sigma < 0$, we have the embedding $L^{\frac{2n}{(n-2s)(\alpha+1)}} \hookrightarrow \dot{B}_{p', 2}^\sigma$ (or $L^{\frac{2n}{(n-2s)(\alpha+1)}} \hookrightarrow \dot{H}_{p'}^\sigma$), where $p' = \frac{2n}{2\sigma + (n-2s)(\alpha+1)}$, and then applying Hölder's inequality and Sobolev embedding $\dot{H}^s \hookrightarrow L^{\frac{2n}{n-2s}}$, one can get

$$\begin{aligned} \|f(u) - f(v)\|_{\dot{B}_{p', 2}^\sigma} &\lesssim \|f(u) - f(v)\|_{L^{\frac{2n}{(n-2s)(\alpha+1)}}} \\ &\lesssim \|(|u|^\alpha + |v|^\alpha)|u - v|\|_{L^{\frac{2n}{(n-2s)(\alpha+1)}}} \lesssim (\|u\|_{\dot{H}^s}^\alpha + \|v\|_{\dot{H}^s}^\alpha) \|u - v\|_{\dot{H}^s}. \end{aligned} \quad (3.11)$$

Let (λ, p) be an $\frac{n}{2}$ -acceptable pair, and it is easy to verify that if $\frac{\frac{n}{2} + \frac{2}{n} - 2}{n-1} < \frac{1}{\rho} < \frac{n-2}{2(n-1)}$, then we can choose γ and λ such that (γ, ρ) and (λ, p) satisfy the conditions (2.3) and (2.6)-(2.8). Then by the sharp case of Lemma 2.1, finite time T and (3.11), it follows that

$$\|u - v\|_{L^\gamma(0, T; \dot{B}_{\rho, 2}^\sigma)} \lesssim \|f(u) - f(v)\|_{L^\lambda(0, T; \dot{B}_{p', 2}^\sigma)} < +\infty. \quad (3.12)$$

Therefore, for any $0 < \alpha < \min\left\{\frac{4}{n-2s}, \frac{n+2s}{n-2s}\right\}$, we have that $u - v$ belongs to the space $L^\gamma(0, T; \dot{B}_{\rho, 2}^\sigma)$ (or $L^\gamma(0, T; \dot{H}_\rho^\sigma)$).

The rest is the same as what we did in Part 3.1, except the selection of ρ : $\frac{1}{\rho} = \frac{\sigma}{n} + \frac{1}{2} - \frac{s}{n} + \frac{n-2s}{2n}\alpha - \frac{2}{n}$. Then after a series of calculations, we can show that σ has to satisfy the conditions

$$\max\left\{-s, s + \frac{3n-4}{2n-2} - \frac{3n-4}{4n-4}(n-2s)\alpha\right\} \leq \sigma < \min\left\{0, s + \frac{3n-4}{2n-2} - \frac{n-2s}{2}\alpha\right\}. \quad (3.13)$$

Obviously, σ is existent when $\alpha \in [\frac{2+[(n-1)4s/(3n-4)]}{n-2s}, \frac{4s+4-n/(n-1)}{n-2s})$. In view of (3.10), we have

$$\frac{2+[(n-1)4s/(3n-4)]}{n-2s} < (2+8s(1-\frac{1}{n}))(n-2s) < \frac{4s+4-n/(n-1)}{n-2s}.$$

Therefore, unconditional uniqueness is proved under the assumptions

$$\max\{(2+8s(1-\frac{1}{n}))(n-2s), 1\} < \alpha < \min\{\frac{4s+4-n/(n-1)}{n-2s}, \frac{4}{n-2s}, \frac{n+2s}{n-2s}\}.$$

In summary, by the results of Part 3.1 and Part 3.2, if α , s and n satisfy the conditions

$$\begin{cases} \max\{1, \frac{2s}{n-2s}\} \leq \alpha < \min\{\frac{4}{n-2s}, \frac{n+2s}{n-2s}, \frac{4s+4-n/(n-1)}{n-2s}\}, \\ 0 < s < 1 \text{ and } n = 3, 4, 5, \end{cases} \quad (3.14)$$

we have unconditional uniqueness of (NLS), which conclude the proof of Theorem 1.1.

3.3. The proof of theorem 1.2. In this subsection, we give a sketch of proof of Theorem 1.2. The proof proceeds follows that of Theorem 1.1. Instead of $L^\gamma(0, T; \dot{B}_{\rho, 2}^\sigma)$ there, we use the space $L^\gamma(0, T; \dot{H}_\rho^\sigma)$ for $\alpha < 1$, and Lemma 2.5 in Sobolev version. The main difference is how to get a similar estimate as (3.5), now we should consider $\int_0^1 \|f'(v + \theta(u - v))\|_{\dot{H}_{p_1}^{-\sigma}} d\theta$.

Since $f' \in C^{0, \alpha}$, Lemma 2.4 and Sobolev embedding $\dot{H}^s \hookrightarrow L^{\frac{2n}{n-2s}}$ lead to

$$\begin{aligned} \int_0^1 \|f'(v + \theta(u - v))\|_{\dot{H}_{p_1}^{-\sigma}} d\theta &\lesssim \int_0^1 \|v + \theta(u - v)\|_{L^{\frac{2n}{n-2s}}}^{\alpha + \frac{\sigma}{s}} \|v + \theta(u - v)\|_{\dot{H}^s}^{\frac{-\sigma}{s}} d\theta \\ &\lesssim (\|u\|_{\dot{H}^s} + \|v\|_{\dot{H}^s})^\alpha. \end{aligned} \quad (3.15)$$

Similarly, when ρ is chosen as $\frac{1}{\rho} = \frac{\sigma}{n} + \frac{1}{2} - \frac{s}{n}$, we can summarize the conditions imposed on parameters q, r, γ, ρ , and σ :

- (1) the choices of ρ and r : $\frac{1}{\rho} = \frac{\sigma}{n} + \frac{1}{2} - \frac{s}{n}$, $\frac{1}{r} = \frac{1}{2} - \frac{\sigma}{n} + \frac{s}{n} - \frac{n-2s}{2n}\alpha$;
- (2) (γ, ρ) , (q, r) being $\frac{n}{2}$ -acceptable pairs;
- (3) (γ, ρ) , (q, r) satisfying the conditions (2.3)-(2.5);
- (4) conditions on σ and r for the validity of Lemma 2.5,

$$-1 < \sigma < 0, \quad 0 < \frac{1}{r} + \frac{\sigma}{n} \leq 1,$$

where the second one is equivalent to $\frac{2s-n}{n-2s} \leq \alpha < \frac{n+2s}{n-2s}$;

- (5) condition on σ for the validity of Lemma 2.4,

$$-\alpha s < \sigma.$$

These conditions still infer the conditions on σ ,

$$\max\{s - \frac{n-2s}{2}\alpha, s - \frac{n}{2}, -\alpha s\} < \sigma < 0, \quad (3.16)$$

$$s - \frac{n}{2(n-1)} - \frac{(n-2)(n-2s)}{4(n-1)}\alpha \leq \sigma \leq s + \frac{n}{2(n-1)} - \frac{n(n-2s)}{4(n-1)}\alpha. \quad (3.17)$$

Therefore, in order to establish unconditional uniqueness of (NLS), we have to find q, r, γ, ρ and σ fulfill all of the restrictions. Through a series of calculations, these parameters can be chosen if s, α and n satisfy one of the following conditions:

- when $n = 3$,

$$\begin{cases} \frac{2s}{n-2s} < \alpha < 1, & \text{if } s \geq \frac{9}{14}, \\ \frac{2s}{n-2s} < \alpha < \min\{1, \frac{2+4s(1-\frac{1}{n})}{(n-2s)-4s(1-\frac{1}{n})}\}, & \text{if } s \leq \frac{9}{14}, \end{cases} \quad (3.18)$$

- when $n = 4$,

$$\begin{cases} \frac{2s}{n-2s} < \alpha < 1, & \text{if } s \geq \frac{4}{5}, \\ \frac{2s}{n-2s} < \alpha < \min\{1, \frac{2+4s(1-\frac{1}{n})}{(n-2s)-4s(1-\frac{1}{n})}\}, & \text{if } s \leq \frac{4}{5}, \end{cases} \quad (3.19)$$

- when $n = 5$,

$$\begin{cases} \frac{2s}{n-2s} < \alpha < 1, & \text{if } s \geq \frac{25}{26}, \\ \frac{2s}{n-2s} < \alpha < \min\{1, \frac{4}{n-2s}, \frac{2+4s(1-\frac{1}{n})}{(n-2s)-4s(1-\frac{1}{n})}\}, & \text{if } s \leq \frac{25}{26}, \end{cases} \quad (3.20)$$

- when $n \geq 6$,

$$\frac{2s}{n-2s} < \alpha < \min\left\{\frac{4}{n-2s}, \frac{2+4s(1-\frac{1}{n})}{(n-2s)-4s(1-\frac{1}{n})}\right\}. \quad (3.21)$$

Next, we consider the case $\frac{2+4s(1-\frac{1}{n})}{(n-2s)-4s(1-\frac{1}{n})} \leq \alpha < \min\{1, \frac{4}{n-2s}, \frac{n+2s}{n-2s}\}$. As showed in Part 3.2, $u - v$ is in the space $L^\gamma(0, T; \dot{H}_\rho^\sigma)$ with $\frac{1}{\rho} = \frac{\sigma}{n} + \frac{1}{2} - \frac{s}{n} + \frac{n-2s}{2n}\alpha - \frac{2}{n}$. Therefore, we invoke the property and the same argument as above to reduce the restrictions of σ :

$$\max\{s + 2 - (n-2s)\alpha, -\alpha s\} < \sigma < s + 2 - \frac{n-2s}{2}\alpha - \frac{n}{2(n-1)}, \quad (3.22)$$

$$s + 2 - \frac{n}{2(n-1)} - \frac{3n/4-1}{n-1}(n-2s)\alpha < \sigma < 0. \quad (3.23)$$

In order to ask that σ satisfies the conditions (3.22)-(3.23), the following relationship has to be satisfied

$$\frac{2ns + 3n - 2s - 4}{(3n/2 - 2)(n - 2s)} < \alpha < \frac{2s + 4 - \frac{n}{n-1}}{n - 4s}.$$

Noticed the prior assumption $\frac{2+4s(1-\frac{1}{n})}{(n-2s)-4s(1-\frac{1}{n})} \leq \alpha < \min\{1, \frac{4}{n-2s}, \frac{n+2s}{n-2s}\}$, we can see

$$\frac{2ns + 3n - 2s - 4}{(3n/2 - 2)(n - 2s)} < \frac{2 + 4s(1-\frac{1}{n})}{(n-2s) - 4s(1-\frac{1}{n})} < \frac{2s + 4 - \frac{n}{n-1}}{n - 4s}.$$

Therefore, we have shown unconditional uniqueness under the following conditions:

$$\frac{2 + 4s(1-\frac{1}{n})}{(n-2s) - 4s(1-\frac{1}{n})} \leq \alpha < \min\left\{1, \frac{4}{n-2s}, \frac{n+2s}{n-2s}, \frac{2s + 4 - \frac{n}{n-1}}{n - 4s}\right\}.$$

In summary, by the results of above, we have unconditional uniqueness of (NLS) if α, s and n satisfy the following conditions:

- when $n = 3$,

$$\begin{cases} \frac{2s}{3-2s} < \alpha < 1, & \text{if } s > \frac{3}{4}, \\ \frac{2s}{3-2s} < \alpha < \min\{1, \frac{2s+\frac{5}{2}}{3-4s}\}, & \text{if } s \leq \frac{3}{4}, \end{cases} \quad (3.24)$$

- when $n \geq 4$,

$$\frac{2s}{n-2s} < \alpha < \min\left\{1, \frac{4}{n-2s}, \frac{2s + 4 - \frac{n}{n-1}}{n - 4s}\right\}. \quad (3.25)$$

Hence, we finish the proof of Theorem 1.2.

4. THE PROOF OF THEOREMS 1.5 AND 1.6

In this section, we give the proof of Theorems 1.5 and 1.6. By the argument of the subcritical case, one can find σ is a function with respect to parameters α , s and n . If we consider the critical case, α can be determined by s . So for some special dimensions, we can fix the choice of σ .

4.1. **The case of $n = 2$, $\alpha = \frac{2+2s}{2-2s}$ and $0 < s < 1$.**

4.1.1. **The case of $n = 2$, $\alpha = \frac{2+2s}{2-2s}$ and $0 < s < \frac{1}{2}$.** In this situation, we select $\sigma = -s + \varepsilon$, $\frac{1}{\lambda} = \frac{1}{2} + \frac{\varepsilon}{2}$, $(\frac{1}{a}, \frac{1}{b}) = (\frac{s}{2}, \frac{1}{2} - s)$, where ε is a sufficiently small constant such that $0 < \varepsilon < s$.

By Sobolev Embedding $L^1 \hookrightarrow \dot{H}^{\sigma}_{\frac{2}{2+\sigma}}$, $\dot{H}^s \hookrightarrow L^{\frac{4}{2-2s}}$, Hölder inequality and $|f(u) - f(v)| \lesssim (|u|^\alpha + |v|^\alpha)(u - v)$, we can see

$$\begin{aligned} \|f(u) - f(v)\|_{\dot{H}^{\sigma}_{\frac{2}{2+\sigma}}} &\lesssim \|f(u) - f(v)\|_{L^1} \\ &\lesssim (\|u\|_{L^{\frac{4}{2-2s}}}^\alpha + \|v\|_{L^{\frac{4}{2-2s}}}^\alpha) \|u - v\|_{L^{\frac{4}{2-2s}}} \lesssim \|u, v\|_{\dot{H}^s}^{\alpha+1}. \end{aligned}$$

It is easy to see that (a, b) and $(\lambda, -\frac{2}{\sigma})$ are $\frac{n}{2}$ -acceptable pairs, then by Lemma 2.1, we can get

$$\|u - v\|_{L^a(0, T; \dot{H}_b^\sigma)} \lesssim \|f(u) - f(v)\|_{L^{\lambda'}(0, T; \dot{H}^{\sigma}_{\frac{2}{2+\sigma}})} \lesssim T^{\frac{1}{\lambda'}} \|u, v\|_{L^\infty(0, T; \dot{H}^s)}^{\alpha+1}, \quad (4.1)$$

where (a, b) and $(\lambda, -\frac{2}{\sigma})$ satisfy the conditions of $n = 2$ in Lemma 2.1. From (4.1), one can find if $u, v \in L^\infty(0, T; \dot{H}^s)$, then $u - v \in L^a(0, T; \dot{H}_b^\sigma)$.

Furthermore, we choose $(q, r) = (2, \frac{2}{s})$ which is an $\frac{n}{2}$ -acceptable pair. By a simple calculation, one can see (a, b) and (q, r) satisfying the conditions of Lemma 2.1, then it follows that

$$\|u - v\|_{L^a(0, T; \dot{H}_b^\sigma)} \lesssim \|f(u) - f(v)\|_{L^{q'}(0, T; \dot{H}_r^\sigma)}. \quad (4.2)$$

If we select $\frac{1}{p_1} = \frac{1}{2} + s - \frac{\varepsilon}{2}$ and $\frac{1}{p_3} = \frac{1+s}{2}$, then r, b, p_1 and p_3 satisfy the conditions of bilinear estimate Lemma 2.5. So it follows that

$$\begin{aligned} &\|f(u) - f(v)\|_{\dot{H}_r^\sigma} \\ &\lesssim \|u - v\|_{\dot{H}_b^\sigma} \left(\int_0^1 \|f'(\theta u + (1 - \theta)v)\|_{\dot{H}_{p_1}^{-\sigma}} d\theta + \int_0^1 \|f'(\theta u + (1 - \theta)v)\|_{L^{p_3}} d\theta \right). \end{aligned} \quad (4.3)$$

For the second term of the right hand side of (4.3), by Sobolev Embedding $\dot{H}^s \hookrightarrow L^{\frac{4}{2-2s}}$, we can obtain

$$\int_0^1 \|f'(\theta u + (1 - \theta)v)\|_{L^{p_3}} d\theta \lesssim (\|u\|_{L^{p_3\alpha}} + \|v\|_{L^{p_3\alpha}})^\alpha \lesssim (\|u\|_{\dot{H}^s} + \|v\|_{\dot{H}^s})^\alpha. \quad (4.4)$$

For the first term of the right hand side of (4.3), by Lemma 2.3 and $\dot{H}^s \hookrightarrow \dot{H}_l^{-\sigma}$, where $\frac{1}{l} = \frac{1}{2} - \frac{\varepsilon}{2}$, we can obtain

$$\begin{aligned} \int_0^1 \|f'(\theta u + (1 - \theta)v)\|_{\dot{H}_{p_1}^{-\sigma}} d\theta &\lesssim \left(\|u\|_{L^{\frac{4}{2-2s}}} + \|v\|_{L^{\frac{4}{2-2s}}} \right)^{\alpha-1} (\|u\|_{\dot{H}_l^{-\sigma}} + \|v\|_{\dot{H}_l^{-\sigma}}) \\ &\lesssim (\|u\|_{\dot{H}^s} + \|v\|_{\dot{H}^s})^\alpha. \end{aligned} \quad (4.5)$$

In conclusion, by (4.2)-(4.5) and Hölder inequality on time, we can obtain

$$\|u - v\|_{L^a(0,T;\dot{H}_b^\sigma)} \lesssim T^{\frac{1}{2}-\frac{1}{a}} \left(\|u\|_{L^\infty(0,T;\dot{H}^s)} + \|v\|_{L^\infty(0,T;\dot{H}^s)} \right)^\alpha \|u - v\|_{L^a(0,T;\dot{H}_b^\sigma)}, \quad (4.6)$$

which shows the unconditional uniqueness if T sufficiently small.

4.1.2. The case of $n = 2$, $\alpha = \frac{2+2s}{2-2s}$ and $\frac{1}{2} \leq s < 1$. In this case, the conclusion follows from the same argument as above, but the choice of

$$\begin{aligned} \sigma &= s - 1 + 2\varepsilon, & \frac{1}{a} &= \frac{1}{2} - \frac{\varepsilon}{2}, & \frac{1}{b} &= \frac{\varepsilon}{2}, \\ \frac{1}{\lambda} &= \frac{s}{2} + \varepsilon, & \frac{1}{q} &= \frac{s}{2} + \frac{\varepsilon}{2}, & \frac{1}{r} &= \frac{1}{2} - \frac{\varepsilon}{2} - \frac{s}{2}, \\ \frac{1}{p_1} &= 1 - \varepsilon, & \frac{1}{p_3} &= \frac{1}{2} + \frac{s}{2}, \end{aligned}$$

where ε is a sufficiently small constant such that $0 < \varepsilon < \frac{1}{2} - \frac{s}{2}$.

4.2. The case of $n = 3$, $\alpha = \min\{\frac{3+2s}{3-2s}, \frac{4}{3-2s}\}$.

4.2.1. The case of $n = 3$, $\alpha = \frac{3+2s}{3-2s}$ and $\frac{1}{4} < s < \frac{1}{2}$. Similar to the case of $n = 2$, in this case, to get the conclusion we need to use the non-sharp case of Lemma 2.1 for $n = 3$ and choose

$$\begin{aligned} \sigma &= -s, & \frac{1}{a} &= \frac{1}{2} - \frac{1}{b}, \\ \frac{1}{\lambda} &= 1 - \frac{s}{2} - \frac{1}{2b}, & \frac{1}{q} &= \frac{1}{4} + \frac{s}{2} + \frac{1}{b}, & \frac{1}{r} &= \frac{1}{2} - \frac{1}{b} - \frac{s}{3}, \\ \frac{1}{p_1} &= \frac{1}{2} + \frac{2s}{3}, & \frac{1}{p_3} &= \frac{1}{2} + \frac{s}{3}, \end{aligned}$$

and $\frac{1}{b}$ satisfies $\frac{1}{3} - \frac{s}{3} < \frac{1}{b} < \min\{s, \frac{1}{2} - \frac{2s}{3}\}$.

So far, we have completed the proof of Theorem 1.5.

4.2.2. The case of $n = 3$, $\alpha = \frac{4}{3-2s}$ and $\frac{1}{2} < s < 1$. In this case, we choose $\sigma = s - 1$. By Lemma 2.1 (or classical Strichartz estimates), we can obtain

$$\|u - v\|_{L^2(0,T;\dot{H}_6^\sigma)} + \|u - v\|_{L^4(0,T;\dot{H}_3^\sigma)} \lesssim \|f(u) - f(v)\|_{L^2(0,T;\dot{H}_{6/5}^\sigma)}. \quad (4.7)$$

By Sobolev Embedding $L^{\frac{6}{7-2s}} \hookrightarrow \dot{H}_{6/5}^\sigma$, $\dot{H}^s \hookrightarrow L^{\frac{6}{3-2s}}$ and $|f(u) - f(v)| \lesssim (|u|^\alpha + |v|^\alpha)(u - v)$, we have

$$\|f(u) - f(v)\|_{\dot{H}_{6/5}^\sigma} \lesssim \|u, v\|_{\dot{H}^s}^{\alpha+1},$$

it follows $u - v \in L^2(0, T; \dot{H}_6^\sigma) \cap L^4(0, T; \dot{H}_3^\sigma)$.

We denote that

$$\begin{aligned} f(u) - f(v) &= \left[\int_0^1 f'(\theta u + (1 - \theta)v) d\theta \right] (u - v) \\ &= \left\{ \int_0^1 [P_{\leq N} f'(\theta u + (1 - \theta)v)] d\theta \right\} (u - v) + \left\{ \int_0^1 [P_{> N} f'(\theta u + (1 - \theta)v)] d\theta \right\} (u - v) \\ &:= (I) + (II). \end{aligned} \quad (4.8)$$

For the term (II), by using Lemma 2.5 with $\frac{1}{p_1} = 1 - \frac{s}{3}$ and $\frac{1}{p_3} = \frac{2}{3}$, we can obtain

$$\begin{aligned} \|(II)\|_{L^2(0,T;\dot{H}_{6/5}^\sigma)} &\lesssim \left\{ \left\| \int_0^1 [P_{>N} f'(\theta u + (1-\theta)v)] d\theta \right\|_{L^\infty(0,T;\dot{H}_{p_1}^{-\sigma})} \right. \\ &\quad \left. + \left\| \int_0^1 [P_{>N} f'(\theta u + (1-\theta)v)] d\theta \right\|_{L^\infty(0,T;L^{p_3})} \right\} \|u-v\|_{L^2(0,T;\dot{H}_6^\sigma)} \end{aligned} \quad (4.9)$$

It follows from Lemma 2.3 and the Sobolev Embedding $\dot{H}^s \hookrightarrow L^{\frac{6}{3-2s}}$ that

$$\begin{aligned} &\left\| \int_0^1 [P_{>N} f'(\theta u + (1-\theta)v)] d\theta \right\|_{L^\infty(0,T;\dot{H}_{p_1}^{-\sigma})} \\ &+ \left\| \int_0^1 [P_{>N} f'(\theta u + (1-\theta)v)] d\theta \right\|_{L^\infty(0,T;L^{p_3})} \lesssim \left(\|u\|_{L^\infty(0,T;\dot{H}^s)} + \|v\|_{L^\infty(0,T;\dot{H}^s)} \right)^\alpha \end{aligned} \quad (4.10)$$

Note that $u, v \in C([0, T], \dot{H}^s)$, so we can find a uniform N_0 independent on time such that when $N > N_0$,

$$\begin{aligned} &C \left(\left\| \int_0^1 [P_{>N} f'(\theta u + (1-\theta)v)] d\theta \right\|_{L^\infty(0,T;\dot{H}_{p_1}^{-\sigma})} \right. \\ &\quad \left. + \left\| \int_0^1 [P_{>N} f'(\theta u + (1-\theta)v)] d\theta \right\|_{L^\infty(0,T;L^{p_3})} \right) \leq \frac{1}{2}. \end{aligned} \quad (4.11)$$

For the term (I), by using the same method applied for the proof of Lemma 2.5 and Bernstein inequality, we obtain

$$\begin{aligned} \|(P_{\leq N} f')(u-v)\|_{\dot{H}_{6/5}^\sigma} &\lesssim \|u-v\|_{\dot{H}_3^\sigma} (\|P_{\leq N} f'\|_{\dot{H}_{p_1}^{-\sigma}} + \|P_{\leq N} f'\|_{L^2}) \\ &\lesssim N^{\frac{1}{2}} \|u-v\|_{\dot{H}_3^\sigma} (\|P_{\leq N} f'\|_{\dot{H}_{p_1}^{-\sigma}} + \|P_{\leq N} f'\|_{L^{p_3}}) \end{aligned} \quad (4.12)$$

By Lemma 2.3, the Sobolev Embedding $\dot{H}^s \hookrightarrow L^{\frac{6}{3-2s}}$ and Hölder's inequality on time, it follows that

$$\|(I)\|_{L^2(0,T;\dot{H}_{6/5}^\sigma)} \lesssim T^{\frac{1}{4}} N^{\frac{1}{2}} \left(\|u\|_{L^\infty(0,T;\dot{H}^s)} + \|v\|_{L^\infty(0,T;\dot{H}^s)} \right)^\alpha \|u-v\|_{L^4(0,T;\dot{H}_3^\sigma)}. \quad (4.13)$$

Then by (4.7), (4.13), (4.9) and (4.11), we have

$$\|u-v\|_{L^2(0,T;\dot{H}_6^\sigma)} + \|u-v\|_{L^4(0,T;\dot{H}_3^\sigma)} \leq \frac{1}{2} \|u-v\|_{L^2(0,T;\dot{H}_6^\sigma)} + CT^{\frac{1}{4}} N^{\frac{1}{2}} \|u-v\|_{L^4(0,T;\dot{H}_3^\sigma)}. \quad (4.14)$$

If we choose T small enough so that $CT^{\frac{1}{4}} N^{\frac{1}{2}} < \frac{1}{4}$, then the right hand side can be absorbed by the left hand side, which shows the unconditional uniqueness.

4.3. The case of $n \geq 4$, $\alpha = \frac{4}{n-2s}$.

4.3.1. The boundness of the norm of $u-v$. Suppose (a, b) and $(\lambda, \frac{2n}{n-2\sigma-4+2s})$ are $\frac{n}{2}$ -acceptable pairs. By Sobolev Embedding $L^{\frac{2n}{4+n-2s}} \hookrightarrow \dot{H}^\sigma_{\frac{2n}{2\sigma+4+n-2s}}$, $\dot{H}^s \hookrightarrow L^{\frac{2n}{n-2s}}$, Hölder inequality and $|f(u) - f(v)| \lesssim (|u|^\alpha + |v|^\alpha)(u-v)$, we can get

$$\|f(u) - f(v)\|_{\dot{H}^\sigma_{\frac{2n}{2\sigma+4+n-2s}}} \lesssim \|u, v\|_{\dot{H}^s}^{\alpha+1}. \quad (4.15)$$

If we can show

$$\|u - v\|_{L^a(0,T;\dot{H}_b^\sigma)} \lesssim \|f(u) - f(v)\|_{L^{\lambda'}(0,T;\dot{H}_b^\sigma)}, \quad (4.16)$$

then together with (4.15) and $u, v \in L^\infty(0,T;\dot{H}^s)$, we obtain the boundness of $\|u - v\|_{L^a(0,T;\dot{H}_b^\sigma)}$.

In order that (4.16) holds, by the non-sharp case of Lemma 2.1, the conditions (2.3)-(2.5) have to be fulfilled besides (a, b) and $(\lambda, \frac{2n}{n-2\sigma-4+2s})$ being $\frac{n}{2}$ -acceptable pairs.

According to the computation, if $s-2 < \sigma < \min\{0, s - \frac{n}{2(n-1)}\}$, we can find (a, b) and $(\lambda, \frac{2n}{n-2\sigma-4+2s})$ that satisfy all above conditions. Here we list the restrictions on b , which are useful for the following estimates

$$\begin{cases} \frac{2\sigma+n-2s}{2n} < \frac{1}{b} < \frac{1}{2}, \\ \frac{(n-2)(n-2\sigma-4+2s)}{2n^2} \leq \frac{1}{b} \leq \frac{n-2\sigma-4+2s}{2(n-2)}. \end{cases} \quad (4.17)$$

4.3.2. The case of $n = 4, 5$, $\alpha = \frac{4}{n-2s} \geq 1$ and $\frac{n}{4(n-1)} < s < 1$. Suppose (γ, ρ) is an $\frac{n}{2}$ -acceptable pair with $\frac{1}{\rho} = \frac{\sigma}{n} + \frac{1}{2} - \frac{s}{n}$, and σ satisfies the conditions

$$-s \leq \sigma < 0, \quad s - \frac{3n-4}{2(n-1)} < \sigma < s - \frac{n}{2(n-1)}. \quad (4.18)$$

If the conditions in (4.18) hold, then by Lemma 2.1, we can find (γ, ρ) , (a, b) and λ such that

$$\|u - v\|_{L^\gamma(0,T;\dot{H}_\rho^\sigma)} + \|u - v\|_{L^a(0,T;\dot{H}_b^\sigma)} \lesssim \|f(u) - f(v)\|_{L^{\lambda'}(0,T;\dot{H}_b^\sigma)}. \quad (4.19)$$

By the condition (2.3), one can see $\frac{1}{\gamma} + \frac{1}{\lambda} = \frac{n}{2}(1 - \frac{1}{\rho} - \frac{n-2\sigma-4+2s}{2n}) = 1$.

We denote that

$$\begin{aligned} f(u) - f(v) &= \left[\int_0^1 f'(\theta u + (1-\theta)v) d\theta \right] (u - v) \\ &= \left\{ \int_0^1 [P_{\leq N} f'(\theta u + (1-\theta)v)] d\theta \right\} (u - v) + \left\{ \int_0^1 [P_{> N} f'(\theta u + (1-\theta)v)] d\theta \right\} (u - v) \\ &:= (I) + (II). \end{aligned} \quad (4.20)$$

For the term (II), by using Lemma 2.5 with $\frac{1}{p_1} = \frac{2}{n} - \frac{\sigma}{n}$ and $\frac{1}{p_3} = \frac{2}{n}$ and (4.18), we can obtain

$$\begin{aligned} \|(II)\|_{L^{\lambda'}(0,T;\dot{H}_b^\sigma)} &\lesssim \left\{ \left\| \int_0^1 [P_{> N} f'(\theta u + (1-\theta)v)] d\theta \right\|_{L^\infty(0,T;\dot{H}_{p_1}^{-\sigma})} \right. \\ &\quad \left. + \left\| \int_0^1 [P_{> N} f'(\theta u + (1-\theta)v)] d\theta \right\|_{L^\infty(0,T;L^{p_3})} \right\} \|u - v\|_{L^\gamma(0,T;\dot{H}_\rho^\sigma)}. \end{aligned} \quad (4.21)$$

Lemma 2.3, (4.18), Sobolev Embedding $\dot{H}^s \hookrightarrow L^{\frac{2n}{n-2s}}$ and $\dot{H}^s \hookrightarrow \dot{H}^{-\sigma}_{\frac{2n}{n-2(s+\sigma)}}$ deduce that

$$\begin{aligned} & \left\| \int_0^1 [P_{>N} f'(\theta u + (1-\theta)v)] d\theta \right\|_{L^\infty(0,T;\dot{H}_{p_1}^{-\sigma})} \\ & + \left\| \int_0^1 [P_{>N} f'(\theta u + (1-\theta)v)] d\theta \right\|_{L^\infty(0,T;L^{p_3})} \lesssim \left(\|u\|_{L^\infty(0,T;\dot{H}^s)} + \|v\|_{L^\infty(0,T;\dot{H}^s)} \right)^\alpha \end{aligned} \quad (4.22)$$

Since $u, v \in C([0, T], \dot{H}^s)$, then we can find a uniform N_0 independent on time such that when $N > N_0$,

$$\begin{aligned} & C \left(\left\| \int_0^1 [P_{>N} f'(\theta u + (1-\theta)v)] d\theta \right\|_{L^\infty(0,T;\dot{H}_{p_1}^{-\sigma})} \right. \\ & \quad \left. + \left\| \int_0^1 [P_{>N} f'(\theta u + (1-\theta)v)] d\theta \right\|_{L^\infty(0,T;L^{p_3})} \right) \leq \frac{1}{2}. \end{aligned} \quad (4.23)$$

For the term (I), by using the same method applied for the proof of Lemma 2.5, we have

$$\|(P_{\leq N} f')(u - v)\|_{\dot{H}_{r'}^\sigma} \lesssim \|u - v\|_{\dot{H}_b^\sigma} (\|P_{\leq N} f'\|_{\dot{H}_{x_1}^{-\sigma}} + \|P_{\leq N} f'\|_{L^{x_2}}), \quad (4.24)$$

where $\frac{1}{x_1} = \frac{1}{2} - \frac{s}{n} + \frac{2}{n} - \frac{1}{b}$ and $\frac{1}{x_2} = \frac{1}{2} + \frac{\sigma}{n} - \frac{s}{n} + \frac{2}{n} - \frac{1}{b}$.

If σ satisfies (4.18), then the conditions on b in (4.17) hold. So by Bernstein's inequality, one has

$$\|P_{\leq N} f'\|_{\dot{H}_{x_1}^{-\sigma}} \lesssim N^{n(\frac{1}{b} - \frac{2\sigma-2s+n}{2n})} \|P_{\leq N} f'\|_{\dot{H}_{p_1}^{-\sigma}}; \quad (4.25)$$

$$\|P_{\leq N} f'\|_{L^{x_2}} \lesssim N^{n(\frac{1}{b} - \frac{2\sigma-2s+n}{2n})} \|P_{\leq N} f'\|_{L^{p_3}}. \quad (4.26)$$

By Lemma 2.3, (4.18), Sobolev Embedding $\dot{H}^s \hookrightarrow L^{\frac{2n}{n-2s}}$, $\dot{H}^s \hookrightarrow \dot{H}^{-\sigma}_{\frac{2n}{n-2(s+\sigma)}}$ and Hölder inequality on time, we have

$$\begin{aligned} & \|(I)\|_{L^{\lambda'}(0,T;\dot{H}^{\sigma}_{\frac{2n}{2\sigma+4+n-2s}})} \\ & \leq CT^{1-\frac{1}{\lambda}-\frac{1}{a}} N^{n(\frac{1}{b} - \frac{2\sigma-2s+n}{2n})} \left(\|u\|_{L^\infty(0,T;\dot{H}^s)} + \|v\|_{L^\infty(0,T;\dot{H}^s)} \right)^\alpha \|u - v\|_{L^a(0,T;\dot{H}_b^\sigma)} \\ & \leq \frac{1}{4} \|u - v\|_{L^a(0,T;\dot{H}_b^\sigma)}, \end{aligned} \quad (4.27)$$

if T is small enough such that

$$CT^{1-\frac{1}{\lambda}-\frac{1}{a}} N^{n(\frac{1}{b} - \frac{2\sigma-2s+n}{2n})} \left(\|u\|_{L^\infty(0,T;\dot{H}^s)} + \|v\|_{L^\infty(0,T;\dot{H}^s)} \right)^\alpha < \frac{1}{4}.$$

In conclusion, by (4.19), (4.21), (4.23) and (4.27), we can have

$$\|u - v\|_{L^\gamma(0,T;\dot{H}_\rho^\sigma)} + \|u - v\|_{L^a(0,T;\dot{H}_b^\sigma)} < \frac{3}{4} (\|u - v\|_{L^\gamma(0,T;\dot{H}_\rho^\sigma)} + \|u - v\|_{L^a(0,T;\dot{H}_b^\sigma)}), \quad (4.28)$$

which shows the unconditional uniqueness.

4.3.3. **The case of $n \geq 5$, $\alpha = \frac{4}{n-2s} < 1$ and $s_0 < s < 1$.** The proof is similar to that for $n = 4, 5$ with $\alpha \geq 1$, except that we apply Lemma 2.4 instead of Lemma 2.3.

We still suppose (γ, ρ) is a $\frac{n}{2}$ -acceptable pair with $\frac{1}{\rho} = \frac{\sigma}{n} + \frac{1}{2} - \frac{s}{n}$, and σ satisfies the conditions

$$\frac{-4s}{n-2s} < \sigma < 0, \quad s - \frac{3n-4}{2(n-1)} < \sigma < s - \frac{n}{2(n-1)} \quad (4.29)$$

By Lemma 2.1, the conditions in (4.29) can help us to find (γ, ρ) , (a, b) and λ such that

$$\|u - v\|_{L^\gamma(0,T;\dot{H}_\rho^\sigma)} + \|u - v\|_{L^a(0,T;\dot{H}_\rho^\sigma)} \lesssim \|f(u) - f(v)\|_{L^{\lambda'}(0,T;\dot{H}_\rho^\sigma)}, \quad (4.30)$$

where $\frac{1}{\gamma} + \frac{1}{\lambda} = \frac{n}{2}(1 - \frac{1}{\rho} - \frac{n-2\sigma-4+2s}{2n}) = 1$ from the condition (2.3).

We denote that

$$\begin{aligned} f(u) - f(v) &= \left[\int_0^1 f'(\theta u + (1-\theta)v) d\theta \right] (u - v) \\ &= \left\{ \int_0^1 [P_{\leq N} f'(\theta u + (1-\theta)v)] d\theta \right\} (u - v) + \left\{ \int_0^1 [P_{> N} f'(\theta u + (1-\theta)v)] d\theta \right\} (u - v) \\ &:= (I) + (II). \end{aligned} \quad (4.31)$$

For the term (II), from Lemma 2.5 with $\frac{1}{p_1} = \frac{2}{n} - \frac{\sigma}{n}$, and $\frac{1}{p_3} = \frac{2}{n}$, and (4.29), one can get

$$\begin{aligned} \|(II)\|_{L^{\lambda'}(0,T;\dot{H}_\rho^\sigma)} &\lesssim \left\{ \left\| \int_0^1 [P_{> N} f'(\theta u + (1-\theta)v)] d\theta \right\|_{L^\infty(0,T;\dot{H}_{p_1}^{-\sigma})} \right. \\ &\quad \left. + \left\| \int_0^1 [P_{> N} f'(\theta u + (1-\theta)v)] d\theta \right\|_{L^\infty(0,T;L^{p_3})} \right\} \|u - v\|_{L^\gamma(0,T;\dot{H}_\rho^\sigma)} \end{aligned} \quad (4.32)$$

By using Lemma 2.4, (4.29) and Sobolev Embedding $\dot{H}^s \hookrightarrow L^{\frac{2n}{n-2s}}$, we can obtain

$$\begin{aligned} &\left\| \int_0^1 [P_{> N} f'(\theta u + (1-\theta)v)] d\theta \right\|_{L^\infty(0,T;\dot{H}_{p_1}^{-\sigma})} \\ &+ \left\| \int_0^1 [P_{> N} f'(\theta u + (1-\theta)v)] d\theta \right\|_{L^\infty(0,T;L^{p_3})} \lesssim \left(\|u\|_{L^\infty(0,T;\dot{H}^s)} + \|v\|_{L^\infty(0,T;\dot{H}^s)} \right)^\alpha \end{aligned} \quad (4.33)$$

Since $u, v \in C([0, T], \dot{H}^s)$, then one can find a uniform N_0 independent on time such that when $N > N_0$,

$$\begin{aligned} C \left(\left\| \int_0^1 [P_{> N} f'(\theta u + (1-\theta)v)] d\theta \right\|_{L^\infty(0,T;\dot{H}_{p_1}^{-\sigma})} \right. \\ \left. + \left\| \int_0^1 [P_{> N} f'(\theta u + (1-\theta)v)] d\theta \right\|_{L^\infty(0,T;L^{p_3})} \right) \leq \frac{1}{2}. \end{aligned} \quad (4.34)$$

For the term (I), we use the same argument as that in the case of $n = 4, 5$ and $\alpha \geq 1$ to get

$$\begin{aligned}
& \| (I) \|_{L^{\lambda'}(0,T;\dot{H}^{\sigma}_{\frac{2n}{2\sigma+4+n-2s}})} \\
& \leq CT^{1-\frac{1}{\lambda}-\frac{1}{\alpha}} N^{n(\frac{1}{b}-\frac{2\sigma-2s+n}{2n})} \left(\left\| \int_0^1 [P_{>N} f'(\theta u + (1-\theta)v)] d\theta \right\|_{L^\infty(0,T;\dot{H}^{-\sigma}_{p_1})} \right. \\
& \quad \left. + \left\| \int_0^1 [P_{>N} f'(\theta u + (1-\theta)v)] d\theta \right\|_{L^\infty(0,T;L^{p_3})} \right) \|u-v\|_{L^a(0,T;\dot{H}^{\sigma}_b)} \\
& \leq CT^{1-\frac{1}{\lambda}-\frac{1}{\alpha}} N^{n(\frac{1}{b}-\frac{2\sigma-2s+n}{2n})} \left(\|u\|_{L^\infty(0,T;\dot{H}^s)} + \|v\|_{L^\infty(0,T;\dot{H}^s)} \right)^\alpha \|u-v\|_{L^a(0,T;\dot{H}^{\sigma}_b)} \\
& \leq \frac{1}{4} \|u-v\|_{L^a(0,T;\dot{H}^{\sigma}_b)}, \quad (4.35)
\end{aligned}$$

if T is sufficiently small such that

$$CT^{1-\frac{1}{\lambda}-\frac{1}{\alpha}} N^{n(\frac{1}{b}-\frac{2\sigma-2s+n}{2n})} \left(\|u\|_{L^\infty(0,T;\dot{H}^s)} + \|v\|_{L^\infty(0,T;\dot{H}^s)} \right)^\alpha < \frac{1}{4}.$$

In conclusion, by using (4.30), (4.32), (4.35) and (4.34), we have

$$\|u-v\|_{L^\gamma(0,T;\dot{H}^{\sigma}_\rho)} + \|u-v\|_{L^a(0,T;\dot{H}^{\sigma}_b)} < \frac{3}{4} (\|u-v\|_{L^\gamma(0,T;\dot{H}^{\sigma}_\rho)} + \|u-v\|_{L^a(0,T;\dot{H}^{\sigma}_b)}), \quad (4.36)$$

which shows the unconditional uniqueness, and we complete the proof of Theorem 1.6.

ACKNOWLEDGEMENT

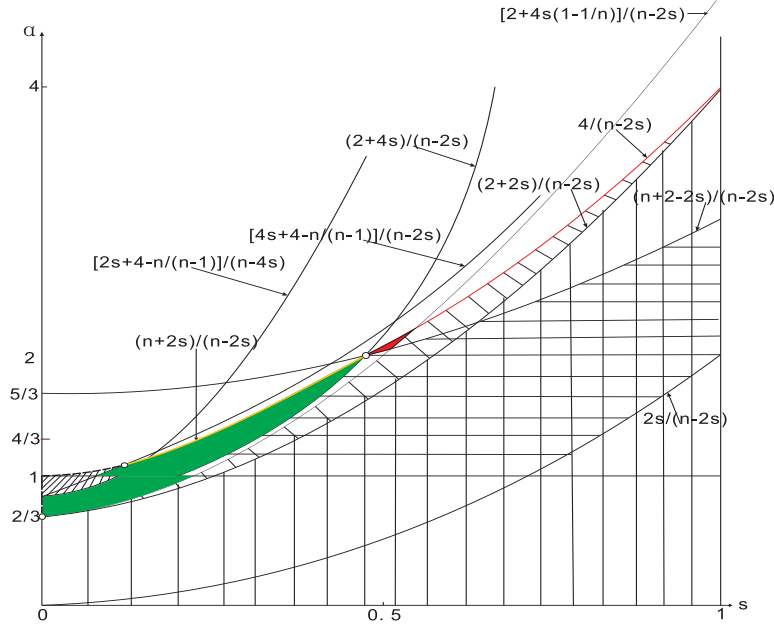
The authors would like to thank Professor T. Cazenave and Professor Y. Tsutsumi for their helpful suggestions. This work is supported partially by NSFC 10871175, 10931007, and Zhejiang NSF of China Z6100217,

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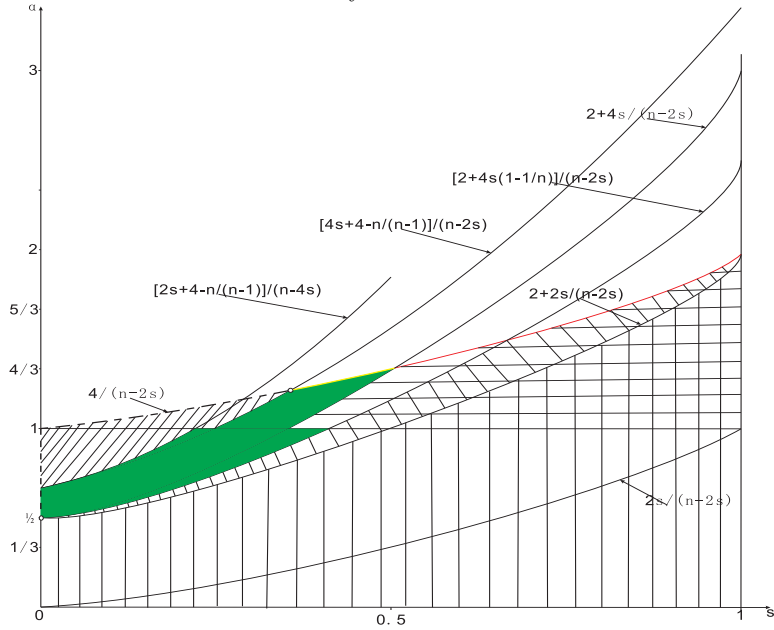
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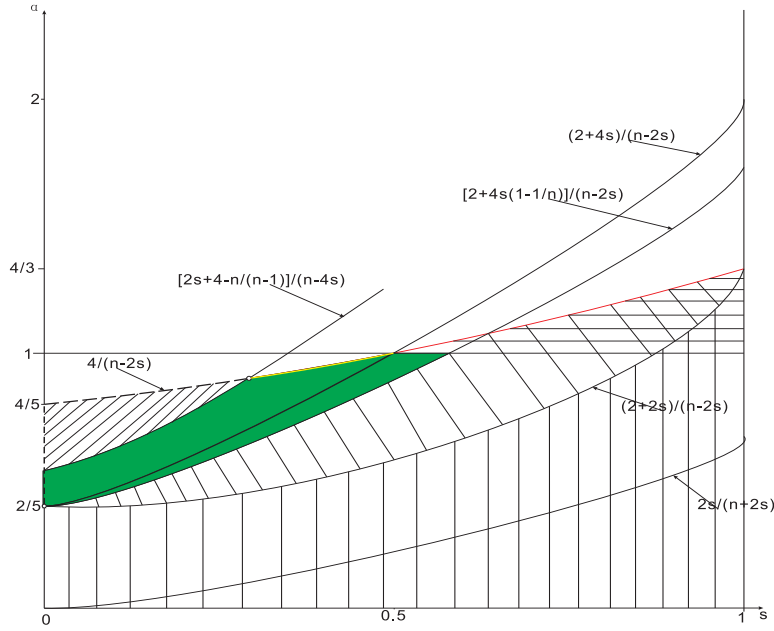
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FIGURE 1. Case $n = 3$

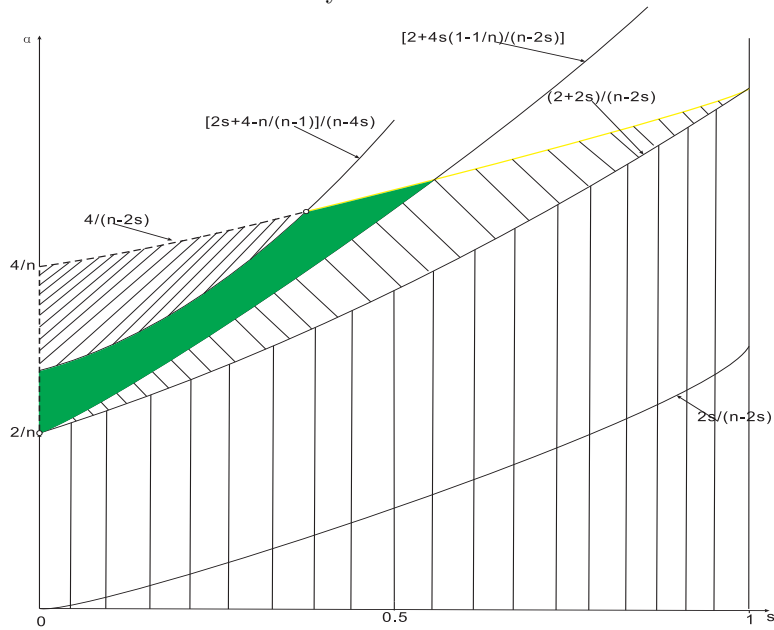
Kato–Vertical route area; Furioli and Terraneo– Horizontal area; Rogers–Oblique line; Win and Tsutsumi–Red color; Open parts–Left slashes and thick dashed lines; Beside to cover the known areas, the new part of our results–Green color for the subcritical case and yellow one for the critical cases.

FIGURE 2. Case $n = 4$

Kato–Vertical route area; Furioli and Terraneo– Horizontal area; Rogers–Oblique line; Win and Tsutsumi–Red color; Open parts–Left slashes and thick dashed lines; Beside to cover the known areas, the new part of our results–Green color for the subcritical case and yellow one for the critical cases.

FIGURE 3. Case $n = 5$

Kato–Vertical area; Furioli and Terraneo– Horizontal area; Rogers–Oblique line;
 Win and Tsutsumi–Red color; Open parts–Left slashes and thick dashed lines;
 Beside to cover the known areas, the new part of our results–Green color for the
 subcritical case and yellow one for the critical cases.

FIGURE 4. Cases $n \geq 6$

Kato–Vertical route area; Rogers–Oblique line; Open parts–Left slashes and thick
 dashed lines; Beside to cover the known areas, the new part of our results–Green
 color for the subcritical case and yellow one for the critical cases.